AIAA JOURNAL
Vol. 40, No. 2, February 2002

Structural Optimization with Frequency Constraints Using the Finite Element Force Method

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A structural optimization algorithm is developed to minimize the weight of structures with truss and beam-type members under single- or multiple-frequency constraints. The cross-sectional areas of the structural members are considered as the design variables. The algorithm proposed combines the finite element technique based on the integrated force method with the mathematical programming technique. The equilibrium matrix is generated automatically using the finite element analysis, and the compatibility matrix is obtained directly using the displacement-deformation relations and the single value decomposition technique. When combining the equilibrium and the compatibility matrices with the force-displacement relations, the frequency eigenvalue equations are obtained with element forces as variables. Three structures, composed of truss and frame-type members, are studied to illustrate the procedure, and the results are compared with the literature. It is shown that, in structural problems with multiple frequency constraints, the analysis procedure (force or displacement method) significantly affects the final optimum design. The structural optimization based on the force method results in a lighter design. The proposed structural optimization method is efficient to analyze and optimize both truss and beam-type structures.

I. Introduction

DESIGN optimization of structures with fundamental or multiple-frequency constraints is extremely useful when improving the dynamic performance of structures. Modifying a particular frequency can significantly improve its overall performance under dynamic external force excitations. Generally, the control of the critical ranges of the natural frequencies is equivalent to the control of the dynamic response in most narrowband forced excitation problems. Structural optimization under frequency constraints gives the ability to a designer to control the selected frequencies in a desired fashion in order to improve the dynamic characteristics of the structure.

The concept of equilibrium of forces and compatibility of deformations is fundamental to analysis methods for solving problems in structural mechanics in general and in structural optimization in particular. The equilibrium equations need to be augmented by the compatibility conditions because the equilibrium equations are indeterminate by nature, and determinacy is achieved by adding the compatibility conditions. Generally, two analytical methods (displacement and force) are available to analyze determinate and indeterminate structures.

Structural analysis and optimization algorithms developed in recent years have generally been based on the displacement method, e.g., Refs. 1–4. The displacement method is an efficient approach for stress-displacement type analysis; however, it presents disadvantages in optimization problems when the number of stress constraints is larger than the number of displacement constraints. Also, in adaptive geometry optimization the displacement method is not the most appropriate analysis method because the element forces are the primary variables.[5,6] For a determinate structure or a not highly redundant structure (the number of redundant elements is lower than the displacement degrees of freedom), analysis using the force method is computationally more efficient than the displacement method. However, the force method has not been very popular among researchers in structural optimization problems because the redundancy analysis required in the force method has not been amenable to computer automation.

In the classical form of the force method, it is very difficult to generate the compatibility conditions. Splitting the given structure into a determinate basis structure and redundant members generates the compatibility in the classical force method. The compatibility conditions are written by establishing the continuity of deformations between redundant members and the basis structure. Navier[7] originally developed this procedure for the analysis of indeterminate trusses. Prior to the 1960s, the basis structure and redundant members were generated manually. In the post-1960s several schemes have been devised to automatically generate redundant members and the basis determinate structure,[8,9] however with limited success. In the integrated force method developed by Patnaik,[10] Patnaik and Joseph,[11] and Patnaik et al.[12] both equilibrium equations and compatibility conditions are solved simultaneously. The generation of compatibility equations is based on extending St. Venant's theory of elasticity strain formulation to discrete structural mechanics and eliminating the displacements in the deformation-displacement relation.

In the present study the integrated force method has been used as an analyzer to optimize both truss and beam structures under frequency constraints. It is intended to investigate the efficiency of the force method in structural optimization of the truss and beam structures under frequency constraints. The equilibrium and compatibility equations are solved simultaneously. A direct method has been developed to generate the compatibility matrix for indeterminate structures. The method is based on the displacement-deformation relation and singular value decomposition (SVD) technique, and there is no need to select consistent redundant members. For both truss and beam structures the equilibrium matrix is generated automatically through the finite element analysis.

In most recent previous works the optimization algorithms were mainly based on the optimality criterion technique. For example, Refs. 13–25 employed optimality criteria methods in minimization of the weight of truss and beam structures. For nonlinear problems the optimality criterion technique is computationally more

Received 8 August 2000; revision received 30 June 2001; accepted for publication 4 July 2001.

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efficient than other mathematical programming techniques.[24,25] A few optimality criterion algorithms[17,19] are based on satisfying one most critical constraint to avoid scaling and calculating a large number of Lagrange multipliers. However, modern optimality criterion algorithms[18,20–23] involve alternatively satisfying multiple constraints (scaling) and Karush–Kuhn–Tuker (KKT) condition (resizing). When the cross-sectional area and principal moment of inertia are nonlinearly related (frame structures), scaling procedures normally used in optimality criterion methods are approximate in nature and scaling itself needs iteration procedure.[23] Moreover, in the case of multiple frequency constraints, determination of the Lagrangian multipliers requires approximations.[22] Considering these, the powerful nonlinear mathematical programming method based on the sequential quadratic programming (SQP) technique has been used as the optimization algorithm in this study to find true optimum solution.

The application and efficiency of the proposed method is illustrated by optimizing the truss and beam structures under single- or multiple-frequency constraints. It is shown that using force or displacement method as an analyzer can affect the final optimum solution in the problems under multiple-frequency constraints. It is illustrated that in problems with multiple-frequency constraints optimization based on the force method can cause lighter design. Using SQP method as the optimizer can result lighter design in comparison to the optimality criterion technique commonly used in the literature.

A short description of the frequency analysis using the integrated force method is presented. This is followed by a description of the SQP algorithm and the derivation of the sensitivity derivatives for the natural frequencies with respect to the cross-sectional areas. Finally, the application of the algorithm is illustrated by optimizing three different truss and beam-type structures.

II. Frequency Analysis Using Integrated Force Method

A discrete finite element structure can be designated as structure (d, f), where d and f are the displacement and force degrees of freedom, respectively. The structure (d, f) has d equilibrium equations and $r = (f - d)$ compatibility conditions. In static problems the equilibrium equations in the displacement formulation can be written as

$$\boldsymbol{KU} = \boldsymbol{P} \tag{1}$$

where $\boldsymbol{K}$ is the system stiffness matrix of the structure (obtained by assembling the stiffness matrices of the individual elements); $\boldsymbol{P}$ is the external applied load vector; and $\boldsymbol{U}$ is the nodal displacement vector. The compatibility conditions have been satisfied implicitly during the generation of the Eq. (1). The equivalent form of the Eq. (1) in the force formulation can be written as[10]

$$\boldsymbol{SF} = \boldsymbol{P}^* \tag{2}$$

which can be obtained through combination of the equilibrium equations as

$$\boldsymbol{QF} = \boldsymbol{P} \tag{3}$$

and compatibility equations as

$$\boldsymbol{C}\Delta = 0 \tag{4}$$

where element deformations can be related to the element forces according to

$$\Delta = \boldsymbol{GF} \tag{5}$$

thus

$$\boldsymbol{S} = \begin{bmatrix} \boldsymbol{Q} \\ \cdots\cdots \\ \boldsymbol{CG} \end{bmatrix}, \qquad \boldsymbol{P}^* = \begin{bmatrix} \boldsymbol{P} \\ \cdots \\ 0 \end{bmatrix} \tag{6}$$

where $\boldsymbol{Q}$, $\boldsymbol{C}$, and $\boldsymbol{G}$ are the $(d \times f)$ equilibrium matrix, $(r \times f)$ compatibility matrix, and $(f \times f)$ flexibility matrix, respectively. The matrices $\boldsymbol{Q}$, $\boldsymbol{C}$, and $\boldsymbol{G}$ are banded, and they have full-row ranks of d, r, and f, respectively; the matrices $\boldsymbol{Q}$ and $\boldsymbol{C}$ depend on the geometry of the structure and are independent of design variables and material properties. For a finite element idealization the generation of the equilibrium matrix $\boldsymbol{Q}$ and the flexibility matrix $\boldsymbol{G}$ are straightforward and can be obtained automatically. However the automatic generation of the compatibility matrix $\boldsymbol{C}$ is a laborious task in the standard force method. In the integrated force method the generation of $\boldsymbol{C}$ is based on the elimination of the d displacement degrees of freedom from f elemental deformations. Here, an efficient method is proposed to derive the compatibility matrix directly. The method is based on the displacement-deformation relations and SVD.

The displacement-deformation relation for discrete structures can be obtained by equating internal strain energy and external work as

$$\tfrac{1}{2}\boldsymbol{F}^T\Delta = \tfrac{1}{2}\boldsymbol{P}^T\boldsymbol{U} \tag{7}$$

where Δ is the deformation vector. By substituting $\boldsymbol{P}$ from Eq. (3) into Eq. (7), we can obtain

$$\tfrac{1}{2}\boldsymbol{F}^T\boldsymbol{Q}^T\boldsymbol{U} = \tfrac{1}{2}\boldsymbol{F}^T\Delta \qquad \text{or} \qquad \tfrac{1}{2}\boldsymbol{F}^T(\boldsymbol{Q}^T\boldsymbol{U} - \Delta) = 0 \tag{8}$$

where Δ is the deformation vector. Because the force vector $\boldsymbol{F}$ is not a null set, we finally obtain the following relation between member deformations and nodal displacements:

$$\Delta = \boldsymbol{Q}^T\boldsymbol{U} \tag{9}$$

Equation (9) relates the f deformations to the d nodal displacement degrees of freedom; therefore, the $r = (f - d)$ compatibility equations can be obtained through elimination of the m nodal displacements from the f deformations. To obtain the compatibility matrix, we can express displacements in terms of deformations using Eq. (9) as

$$\boldsymbol{U} = (\boldsymbol{QQ}^T)^{-1}\boldsymbol{Q}\Delta = (\boldsymbol{Q}^T)^{\text{pinv}}\Delta \tag{10}$$

where the matrix $(\boldsymbol{Q}^T)^{\text{pinv}}$ denotes the Moore–Penrose pseudoinverse of $\boldsymbol{Q}^T$. Substituting displacements $\boldsymbol{U}$ in Eq. (10) into Eq. (9), we have

$$\Delta = \boldsymbol{Q}^T(\boldsymbol{Q}^T)^{\text{pinv}}\Delta \Rightarrow [\boldsymbol{I} - \boldsymbol{Q}^T(\boldsymbol{Q}^T)^{\text{pinv}}]\Delta = 0 \tag{11}$$

or

$$\boldsymbol{B}\Delta = 0 \tag{12}$$

where

$$\boldsymbol{B} = \left[\boldsymbol{I}_n - \boldsymbol{Q}^T(\boldsymbol{Q}^T)^{\text{pinv}}\right] \tag{13}$$

The rank of $(f \times f)$ $\boldsymbol{B}$ matrix is r. It means that the rows of matrix $\boldsymbol{B}$ are dependent on each other. To extract the $(r \times f)$ compatibility matrix $\boldsymbol{C}$ from the matrix $\boldsymbol{B}$, i.e., to reduce the matrix $\boldsymbol{B}$ to matrix $\boldsymbol{C}$, the SVD method is used.[26] Applying SVD to $\boldsymbol{B}$, we obtain

$$\boldsymbol{B} = \boldsymbol{R}\Sigma\boldsymbol{T}^T \tag{14}$$

where $\boldsymbol{R}$ and $\boldsymbol{T}$ are $(f \times f)$ orthogonal matrices and

$$\Sigma = \begin{bmatrix} \Lambda & 0 \\ 0 & 0 \end{bmatrix}_{(f \times f)} \tag{15}$$

with $\Lambda = \text{diag}\{\sigma_1 \;\; \sigma_2 \;\; \cdots \;\; \sigma_r\}$ and $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_r > 0$. It follows that

$$\boldsymbol{B} = \boldsymbol{R}\begin{bmatrix} \boldsymbol{C} \\ 0 \end{bmatrix} \tag{16}$$

Therefore the $(r \times f)$ compatibility matrix $\boldsymbol{C}$ can be represented as

$$\boldsymbol{C} = \Lambda[\boldsymbol{T}_1 \quad \boldsymbol{T}_2 \quad \cdots \quad \boldsymbol{T}_i \quad \cdots \quad \boldsymbol{T}_r]^T \tag{17}$$

where $\boldsymbol{T}_i$ denotes the ith column of matrix $\boldsymbol{T}$.

After obtaining the compatibility matrix $\boldsymbol{C}$, the matrix $\boldsymbol{S}$ can be used to obtain the nodal displacement in relative to the deformations or element forces as

$$\boldsymbol{U} = \boldsymbol{J}\Delta \qquad \text{or} \qquad \boldsymbol{U} = \boldsymbol{JGF} \tag{18}$$

In frequency analysis problems the equations of the motion in the displacement formulation can be written as

$$\boldsymbol{M}\ddot{\boldsymbol{U}} + \boldsymbol{KU} = 0 \tag{19}$$

where $\boldsymbol{M}$ is the stiffness matrix of system and $\ddot{\boldsymbol{U}}$ is acceleration vector. Considering Eq. (18) and noting that $\boldsymbol{KU}$ in the displacement method is equivalent to $\boldsymbol{SF}$ in the force method, Eq. (19) can be written as

$$\boldsymbol{M}^*\ddot{\boldsymbol{F}} + \boldsymbol{SF} = 0 \tag{20}$$

where

$$\boldsymbol{M}^* = \begin{bmatrix} \boldsymbol{MJG} \\ \cdots \\ 0 \end{bmatrix} \tag{21}$$

Equation (20) represents the frequency equation in the framework of the force formulation.

In free vibration analysis it is assumed that element forces are harmonics in time, $\boldsymbol{F} = \bar{\boldsymbol{F}}\sin(\omega t)$, where ω and $\bar{\boldsymbol{F}}$ are radian frequency and force mode shape, respectively. Considering Eq. (19), we can obtain

$$\boldsymbol{S}\bar{\boldsymbol{F}} - \omega^2\boldsymbol{M}^*\bar{\boldsymbol{F}} = 0 \tag{22}$$

The natural frequencies can be obtained easily from Eq. (22) by using an eigenvalue extraction algorithm.

To overcome some computational difficulties during the analysis, the $(n \times n)$ system of equations expressed by Eq. (22) can be reduced to a $(m \times m)$ system by taking advantage of the null matrices. To obtain this, the matrices in Eq. (22) are partitioned into the redundant and basis determinate structure as

$$\begin{bmatrix} \boldsymbol{S}_{\mathrm{dd}} & \boldsymbol{S}_{\mathrm{dr}} \\ \boldsymbol{S}_{\mathrm{rd}} & \boldsymbol{S}_{\mathrm{rr}} \end{bmatrix} \begin{bmatrix} \bar{\boldsymbol{F}}_d \\ \bar{\boldsymbol{F}}_r \end{bmatrix} - \omega^2 \begin{bmatrix} \boldsymbol{M}_r & \boldsymbol{M}_d \\ \cdots\cdots & \\ 0 & \end{bmatrix} \begin{bmatrix} \bar{\boldsymbol{F}}_d \\ \bar{\boldsymbol{F}}_r \end{bmatrix} = 0 \tag{23}$$

or

$$\boldsymbol{S}_{\mathrm{dd}}\bar{\boldsymbol{F}}_d + \boldsymbol{S}_{\mathrm{dr}}\bar{\boldsymbol{F}}_r - \omega^2(\boldsymbol{M}_d\bar{\boldsymbol{F}}_d + \boldsymbol{M}_r\bar{\boldsymbol{F}}_r) = 0, \qquad \boldsymbol{S}_{\mathrm{rd}}\bar{\boldsymbol{F}}_d + \boldsymbol{S}_{\mathrm{rr}}\bar{\boldsymbol{F}}_r = 0 \tag{24}$$

Elimination of $\bar{\boldsymbol{F}}_r$ from the $(n \times n)$ system of equations in Eq. (24) results in the reduced $(m \times m)$ subsystem as

$$\left(\boldsymbol{S}_{\mathrm{dd}} - \boldsymbol{S}_{\mathrm{dr}}\boldsymbol{S}_{\mathrm{rr}}^{-1}\boldsymbol{S}_{\mathrm{rd}}\right)\bar{\boldsymbol{F}}_d - \omega^2\left(\boldsymbol{M}_d - \boldsymbol{M}_r\boldsymbol{S}_{\mathrm{rr}}^{-1}\boldsymbol{S}_{\mathrm{rd}}\right)\bar{\boldsymbol{F}}_d = 0 \tag{25}$$

$$\bar{\boldsymbol{F}}_r = \boldsymbol{S}_{\mathrm{rr}}^{-1}\boldsymbol{S}_{\mathrm{rd}}\bar{\boldsymbol{F}}_d \tag{26}$$

Selection of consistent redundant members ensures the existence of the inverse of $\boldsymbol{S}_{\mathrm{rr}}$. The solution of the reduced eigenvalue problem expressed by Eq. (25) gives all of the eigenvalues, whereas both Eqs. (25) and (26) are used to calculate the force eigenvectors. Once the force mode shapes are known, the displacement mode shapes can be generated by using Eq. (18).

III. Optimization Algorithm

The optimization problem can be defined mathematically as follows.

Minimize structural mass:

$$M(\boldsymbol{A}) = \sum_{i=1}^{n} \rho_i L_i A_i \tag{27}$$

Subject to m natural frequency constraints (behavior constraints):

$$g_j(\boldsymbol{A}) = \omega_j^2 - \tilde{\omega}_j^2 \geq 0, \qquad j = 1, 2, 3, \ldots, m \tag{28}$$

and n side constraints on the design variables

$$A_i - \tilde{A}_i \geq 0, \qquad i = 1, 2, 3, \ldots, n \tag{29}$$

where ρ_i, A_i, L_i and are the density, the cross-sectional area, and length of the ith element, respectively; ω_j and $\tilde{\omega}_j$ are the jth natural frequency and its specified value, respectively; M is the total mass of the structure; $\tilde{A}_i$ is the lower limit on the ith design variable.

In this study the SQP method has been applied to solve the optimization problem discussed. The implementation of the SQP has been done in MATLAB®.[27]

Based on the work by Powell,[28] the method allows you to closely mimic Newton's method for constraint optimization just as is done for unconstraint optimization. SQP is indirectly based on the solution of KKT conditions. Given the problem description in Eqs. (26) and (27), the principal idea is the formulation of a QP subproblem based on a quadratic approximation of the Lagrangian function as follows:

$$\mathrm{Lag}(\boldsymbol{A}, \lambda) = M(\boldsymbol{A}) + \sum_{j=1}^{n} \lambda_j \cdot g_j(\boldsymbol{A}) \tag{30}$$

It is assumed that side constraints in Eq. (29) have been expressed as inequality constraints in Eq. (28). At each major iteration of the SQP method, a QP subproblem is solved. The solution to the QP subproblem produces an estimate of the Lagrange multipliers λ_j $(j = 1, \ldots, n)$, and a search direction vector $\boldsymbol{d}^\nu$ in each iteration ν, which is used to form a new iteration as

$$\boldsymbol{A}^{\nu+1} = \boldsymbol{A}^\nu + \alpha^\nu \boldsymbol{d}^\nu \tag{31}$$

The step length parameter α^ν is determined through a one-dimensional minimization in order to produce a sufficient decrease in the merit function. At the end of the one-dimensional minimization, the Hessian of the Lagrangian, required for the solution of the next positive definite quadratic programming problem, is updated using the Broyden–Fletchet–Goldfarb–Shanno (BFGS) formula as

$$\boldsymbol{H}^{(\nu+1)} = \boldsymbol{H}^{(\nu)} + \frac{\boldsymbol{q}^{(\nu)}\boldsymbol{q}^{T(\nu)}}{\boldsymbol{q}^{T(\nu)}\delta^\nu} - \frac{\boldsymbol{H}^{T(\nu)}\delta^{(\nu)}\delta^{T(\nu)}\boldsymbol{H}^{(\nu)}}{\boldsymbol{d}^{T(\nu)}\boldsymbol{H}^{(\nu)}\boldsymbol{d}^{(\nu)}} \tag{32}$$

where

$$\delta^\nu = \boldsymbol{A}^{\nu+1} - \boldsymbol{A}^\nu \tag{33}$$

$$\boldsymbol{q}^\nu = \nabla\,\mathrm{Lag}(\boldsymbol{A}^{\nu+1}, \lambda^{\nu+1}) - \nabla\,\mathrm{Lag}(\boldsymbol{A}^\nu, \lambda^\nu) \tag{34}$$

and $\boldsymbol{H}$ is the approximation of the Hessian of Lag at $\boldsymbol{A}^{\nu+1}$.

Powell[28] recommends keeping the Hessian positive definite even though it may be positive indefinite at the solution point. If $\boldsymbol{H}^\nu$ is positive definite, then $\boldsymbol{H}^{\nu+1}$ obtained using Eq. (32) is also positive definite if and only if $\boldsymbol{q}^{T(\nu)}\delta^{(\nu)}$ is positive at each iteration. However, when the Lagrangian has a negative curvature at $(\boldsymbol{A}^{\nu+1}, \lambda^{\nu+1})$, $\boldsymbol{q}^{T(\nu)}\delta^{(\nu)}$ is not any more positive. To guarantee that the updated Hessian matrix $\boldsymbol{H}^{\nu+1}$ remains positive definite, Powell suggests replacing $\boldsymbol{q}^\nu$ by

$$\theta\boldsymbol{q}^\nu + (1 - \theta)\boldsymbol{H}^\nu\delta^\nu \tag{35}$$

where $\boldsymbol{q}^\nu$ is given by Eq. (34) and θ is determined by

$$\theta = \begin{cases} 1 & \text{if} \quad \delta^{T\nu}\boldsymbol{q}^\nu \geq 0.2\delta^{T\nu}\boldsymbol{H}\delta^\nu \\ \dfrac{0.8\delta^{T\nu}\boldsymbol{H}\delta^\nu}{0.8\delta^{T\nu}\boldsymbol{H}\delta^\nu - \delta^{T\nu}\boldsymbol{q}^\nu} & \text{if} \quad \delta^{T\nu}\boldsymbol{q}^\nu < 0.2\delta^{T\nu}\boldsymbol{H}\delta^\nu \end{cases} \tag{36}$$

Additional details of the algorithm can be found in Ref. 28. As it is obvious, the required gradient of the Lagrangian function involves the gradient of the objective function and the constraints with respect to the design variables. Providing this information for the optimizer can cause the optimum solution to be caught efficiently and accurately.

In this work the gradient of the objective functions and side constraints are relatively straightforward because they are explicitly dependent on the design variables. The gradient of the behavior constraints (frequency) will be explained in the following section.

IV. Sensitivity of the Frequency Constraints in the Displacement and Force Formulation

A. Displacement Formulation

Similar to the force formulation in Eq. (22), the undamped free vibration in displacement formulation can be explained as

$$(\boldsymbol{K}_E - \mu\boldsymbol{M})\bar{\boldsymbol{U}} = 0 \tag{37}$$

where $\boldsymbol{M}$ is the system mass matrix in the vibration analysis and $\bar{\boldsymbol{U}}$ is the displacement mode shape. $\mu = \omega^2$ is the square of the frequency of the free vibration. Premultiplication of Eq. (37) by $\bar{\boldsymbol{U}}^T$ gives

$$\bar{\boldsymbol{U}}^T(\boldsymbol{K}_E - \mu\boldsymbol{M})\bar{\boldsymbol{U}} = 0 \tag{38}$$

and differentiating Eq. (37) with respect to the typical design variable A_i yields

$$\bar{U}^T\left(\frac{dK_E}{dA_i} - \mu\frac{dM}{dA_i} - M\frac{d\mu}{dA_i}\right)\bar{U} + \frac{d\bar{U}^T}{dA_i}(K_E - \mu M)\bar{U} + \bar{U}^T(K_E - \mu M)\frac{d\bar{U}}{dA_i} = 0 \tag{39}$$

Considering that matrices K_E and M are symmetric and using Eq. (37), the sensitivity of the μ with respect to the typical design variable A_i can be obtained as

$$\frac{d\mu}{dA_i} = \frac{\bar{U}^T[dK_E/dA_i - \mu(dM/dA_i)]\bar{U}}{\bar{U}^T M\bar{U}} \tag{40}$$

The mode shape is often normalized with respect to a symmetric positive definite matrix M such that

$$\bar{U}^T M\bar{U} = 1 \tag{41}$$

thus

$$\frac{d\mu}{dA_i} = \bar{U}^T\left(\frac{dK_E}{dA_i} - \mu\frac{dM}{dA_i}\right)\bar{U} \tag{42}$$

Because the element mass matrix for the beam element is linearly dependent on the design variables (cross-sectional area), dM/dA_i can be found analytically. For the beam elements, with a known relationship between the principal moment of inertia and cross-sectional area, dK_E/dA_i can be computed analytically as well.

B. Force Formulation

The eigenvalue problem in the force formulation can be expressed as in Eq. (22) or alternatively as

$$(S - \mu M^*)\bar{F} = 0 \tag{43}$$

where $\mu = \omega^2$. Because the matrix S is not symmetric, a similar approach as explained in Sec. IV.A cannot be used to obtain the derivative of the eigenvalue μ with respect to the design variables. To overcome this problem, a left-hand eigenvector is used, defined as

$$\bar{F}^{L^T}(S - \mu M^*) = 0 \tag{44}$$

Premultiplication of the Eq. (43) by the transpose of the left-hand eigenvector $\bar{F}^{L^T}$ gives

$$\bar{F}^{L^T}(S - \mu M^*)\bar{F} = 0 \tag{45}$$

Differentiating Eq. (45) with respect to the design variable A_i yields

$$F^{L^T}\left(\frac{dS}{dA_i} - \mu\frac{dM^*}{dA_i} - M^*\frac{d\mu}{dA_i}\right)\bar{F} + \frac{dF^{L^T}}{dA_i}(S - \mu M^*)\bar{F} + \bar{F}^{L^T}(S - \mu M^*)\frac{d\bar{F}}{dA_i} = 0 \tag{46}$$

Considering Eqs. (43) and (44), the sensitivity of the μ with respect to the typical design variable X_i is obtained from Eq. (46):

$$\frac{d\mu}{dA_i} = \frac{\bar{F}^{L^T}\left[dS/dA_i - \mu\left(dM^*/dA_i\right)\right]\bar{F}}{\bar{F}^{L^T}M^*\bar{F}} \tag{47}$$

In this work both the dS^*/dA_i and the dM^*/dA_i matrices have been computed using the central finite difference method.

V. Illustrative Examples

A. 10-Bar Planar Truss

The 10-bar planar truss is shown in the Fig. 1. The material is aluminum with Young's modulus $E = 10^7$ psi (6.89×10^{10} Pa) and density $\rho = 0.1$ lbm/in.^{3} (2770 kg/m^3). The minimum area for all elements was set at 0.1 in.^{2} (0.645 cm^2). At each of the four free nodes, a nonstructural lumped mass of 1000 lbm (2.588 lb-s2/in.) (454 kg) is added.

The number of displacement degrees of freedom is $m = 8$, and the number of force degrees of freedom is $n = 10$. Thus, the number of redundancy is $r = 2$. At each of the four free nodes, a nonstructural lumped mass of 1000 lbm is added. At the initial design all of the cross-sectional areas are 20 in.^{2} (129.03 cm^2), and the initial mass is 8392.94 lbm (3810.39 kg). This problem was investigated by Venkayya and Tischler,[18] as well as by Grandhi and Venkayya,[22] using the optimality criterion and displacement method. First, the structure was designed with a fundamental frequency of 14 Hz alone, using both the displacement and force methods. A minimum weight of 5810.24 lbm (2637.85 kg) was obtained. The number of iterations required using the force method (FM) was lower than that required by the displacement method (DM). The final results for the cross-sectional areas and fundamental frequency are tabulated in Tables 1 and 2, respectively. Venkayya and Tischler[18] have reported a minimum mass of 6665.577 lbm (3026.17 kg), where the optimum design was taken as input to compute the specified natural frequency. A fundamental natural frequency of 14.47 Hz was obtained in the present analysis. A simulation carried out using the solution reported in Ref. 18 as the initial design resulted in a final design, which converged to a lighter solution (5810.24 lbm) (2637.85 kg). The current design was based on a consistent mass matrix. A simulation using a lumped mass matrix resulted in a final design with a minimum mass of 6377.82 lbm (2895.53 kg), suggesting that a lumped mass matrix may have been used by Venkayya and Tischler. To confirm this finding, the optimum result reported in Ref. 18 was used as input to compute the fundamental natural frequency based on the lumped mass matrix. A fundamental natural frequency of 13.96 Hz was obtained.

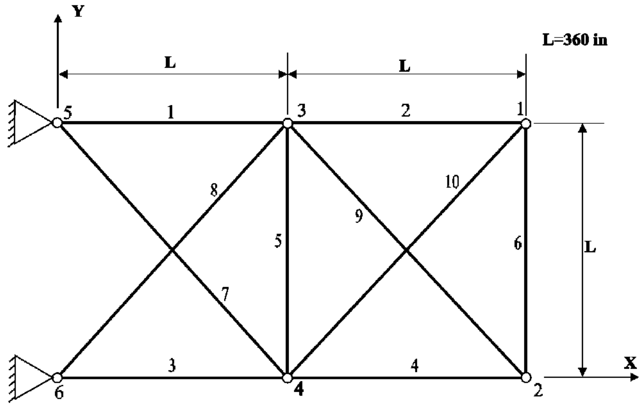

Fig. 1 10-bar planar truss: $L = 360$ in. (914.4 cm).

To demonstrate the application of the algorithm for designing a structure with other specified natural frequencies, the structure was designed for a second natural frequency of 25 Hz. A minimum mass of 1920.52 lbm (871.92 kg) was obtained. Grandhi and Venkayya[22] reported a minimum mass of 2243.8 lbm (1018.69 kg). Here, the optimum design was used as input to compute the second natural frequency, resulting in a solution of 25.37 Hz for the second natural frequency. The final results for the cross-sectional areas and the fundamental frequency are given in the Tables 1 and 2.

Finally, the structure was designed under multiple natural frequency constraints given by $\omega_1 = 7$, $\omega_2 \geq 15$, and $\omega_3 \geq 20$. A minimum mass of 1182.85 lbm (537.01 kg) was obtained. Upon closer inspection the results reveal that the optimum cross-sectional areas for elements 9 and 10 obtained using the FM are different from those using the DM. The optimum masses for both the FM and DM are exactly the same and so are the final natural frequencies. It is inferred that, as the optimizer is very sensitive to the output results from the FM and the DM, a small difference causes the optimizer to select a different path. For this case the number of iterations required by the DM is now lower than that of FM. However, the computational time using the FM is still lower than that of DM. Grandhi and Venkayya[22] have reported a minimum weight of 1308.4 lbm (594.01 kg). The final cross-sectional areas and natural frequencies are given in the Tables 1 and 2. The variation of the optimum mass with the first and second natural frequency limits is shown in Fig. 2. When increasing the fundamental natural frequency limit, the optimum mass increases drastically from 347.91 lbm (157.95 kg) for a fundamental natural frequency limit of 4 Hz to 25,216.7 lbm (11,448.38 kg)

Table 1 Final design for the cross-sectional areas (cm^2) for various frequency constraints for the 10-bar planar truss

	DM			FM		
Element no.	$\omega_1 = 14$	$\omega_2 = 25$	$\omega_1 = 7$ $\omega_2 \geq 15$ $\omega_3 \geq 20$	$\omega_1 = 14$	$\omega_2 = 25$	$\omega_1 = 7$ $\omega_2 \geq 15$ $\omega_3 \geq 20$
1	219.909	48.166	38.619	219.903	48.123	38.245
2	47.916	35.852	18.239	47.916	35.832	9.916
3	219.909	48.194	38.252	219.903	48.200	38.619
4	47.916	35.852	9.910	47.916	35.884	18.232
5	0.645	14.800	4.419	0.645	14.826	4.419
6	0.645	7.632	4.200	0.645	7.632	4.194
7	123.626	41.135	24.110	123.626	41.103	20.097
8	123.626	41.142	20.084	123.626	41.181	24.097
9	54.677	13.200	11.452	54.677	13.200	13.890
10	54.677	13.194	13.897	54.677	13.187	11.4516
Mass, kg	2637.85	871.92	537.01	2637.85	871.92	537.01
No. of iterations	256	1035	637	237	973	705
No. of active constraints	3	1	2	3	1	2
CPU time, s	10.54	40.81	25.96	7.51	28.70	21.62

Table 2 Final design of natural frequencies (Hz) in different frequency constraints for 10-bar truss

		DM			FM		
Frequency no.	Initial design	$\omega_1 = 14$	$\omega_2 = 25$	$\omega_1 = 7$ $\omega_2 \geq 15$ $\omega_3 \geq 20$	$\omega_1 = 14$	$\omega_1 = 25$	$\omega_1 = 7$ $\omega_2 \geq 15$ $\omega_3 \geq 20$
1	11.23	14.00	8.01	7.00	14.00	8.01	7.00
2	33.05	18.01	25.00	17.62	18.01	25.00	17.62
3	36.85	29.40	25.00	20.00	29.40	25.00	20.00
4	68.26	34.55	26.68	20.00	34.55	26.68	20.00
5	75.86	49.36	32.83	28.20	49.36	32.83	28.21
6	85.18	53.11	40.92	31.07	53.11	40.94	31.07
7	85.74	85.10	62.52	47.68	85.10	62.52	47.68
8	103.10	90.41	64.79	52.35	90.41	64.78	52.35

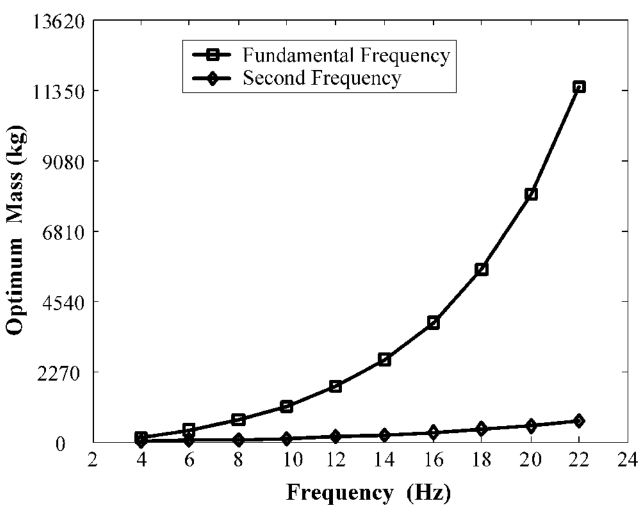

Fig. 2 History of the variation of the optimum mass with respect to the fundamental and second frequencies for the 10-bar planar truss.

for a fundamental natural frequency limit of 22 Hz. However, a less dramatic change is observed for the second natural frequency limit. The optimum mass increases from 46.83 lbm (21.26 kg) for a second natural frequency limit of 4 Hz to 1410.14 lbm (640.20 kg) for second natural frequency limit of 22 Hz.

B. 72-Bar Space Truss

The 72-bar space truss is shown in the Fig. 3. The problem size is relatively large. The material is aluminum with Young's modulus $E = 10^7$ psi (6.89×10^{10} Pa) and density $\rho = 0.1$ lbm/in.^{3} (2770 kg/m^3). The minimum area for all elements was set at 0.1 in.^{2} (0.645 cm^2). The number of displacement degrees of freedom is $m = 48$, and the number of force degrees of freedom is $n = 72$. Therefore, the number of redundancies is $r = 24$. Without linking the design variables, the number of design variables is 72, and the number of constraints is 232. When linking the design variables into 16 groups, the number of design variables becomes 16, and the number of constraints reduces to 64.

At nodes 1–4 a nonstructural lumped mass of 5000 lbm (12.94 lb-s2/in.) (2270 kg) is added. At the initial design all of the cross-sectional areas are 1 in.^{2} (6.45 cm^2), and initial mass is 853.09 lbm (387.30 kg). This problem was investigated by Konzelman[21] using the dual method based on approximation concepts for optimization and the finite element method based on displacement method for analysis. Here, the structure was designed for a fundamental frequency of 4 Hz, using both the displacement and the force methods. A minimum mass of 632.361 lbm (287.09 kg) was obtained. For this example the DM was found to be computationally more efficient than the FM approach. The reason is that SVD technique is not a cheap technique, and when the number of redundancy increases (as in this example) generating the compatibility matrix using SVD becomes computationally expensive. Thus, in optimal design problems with frequency constraints the analysis of not highly redundant structures using the FM is not necessarily more efficient than the DM. The final results for the cross-sectional areas and frequencies are given in Tables 3 and 4, and they are in excellent agreement with those reported by Konzelman,[21] who reported a minimum mass of 632.36 lbm (287.09 kg). Because of the symmetry imposed by the structural geometry and the linking scheme, the eigenvalue corresponding to the fundamental mode of vibration is a repeated eigenvalue of multiciplity two. This means that in the initial and optimum design the first and second modes of vibration have the same natural frequencies. Thus, a small change or deviation in the geometry of the structure can switch the mode of vibration from first to the second mode.

Table 3 Final design for the cross-sectional areas (cm^2) for the various frequency constraints for the 72-bar planar truss

	DM		FM	
Element no.	$\omega_1 = 4$	$\omega_1 = 4$, $\omega_3 \geq 6$	$\omega_1 = 4$	$\omega_1 = 4$, $\omega_3 \geq 6$
1–4	4.717	3.499	4.717	3.499
5–12	5.514	7.932	5.514	7.932
13–16	0.645	0.645	0.645	0.645
17–18	0.645	0.645	0.645	0.645
19–22	11.750	8.056	11.750	8.056
23–30	5.573	8.011	5.573	8.011
31–34	0.645	0.645	0.645	0.645
35–36	0.645	0.645	0.645	0.645
37–40	18.950	12.812	18.950	12.812
41–48	5.607	8.061	5.607	8.061
49–52	0.645	0.645	0.645	0.645
53–54	0.645	0.645	0.645	0.645
55–58	25.935	17.279	25.934	17.279
59–66	5.628	8.088	5.628	8.088
67–70	0.645	0.645	0.645	0.645
71–72	0.645	0.645	0.645	0.645
Mass, kg	287.092	327.605	287.092	327.605
No. of iterations	544	379	510	379
No. of active constraints	9	10	9	10
CPU time, s	283.78	200.37	302.96	227.62

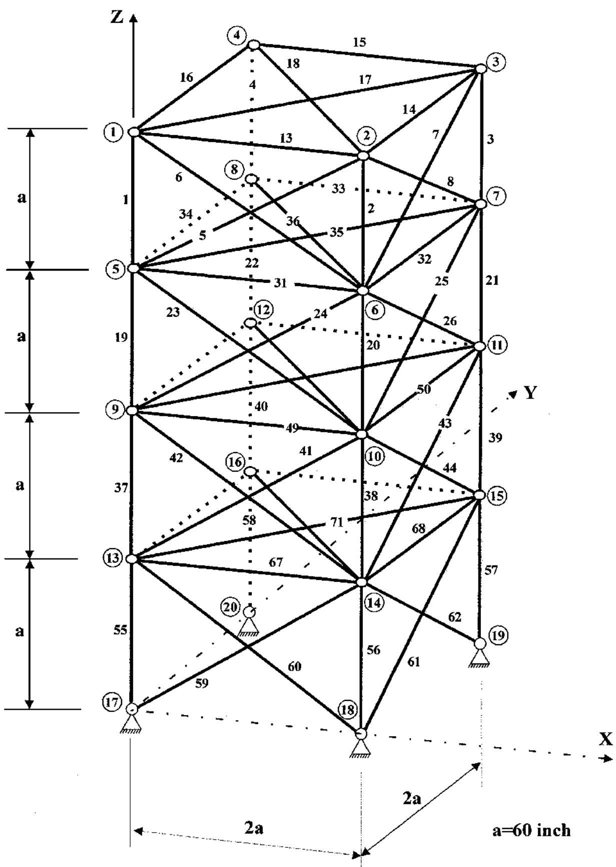

Fig. 3 72-bar space truss: a = 60 in. (152.4 cm).

When the eigenvalues are repeated, the structure becomes extremely sensitive to any change in design variables. Usually the natural frequencies of the first few modes are important, and to separate these eigenvalues in the optimum design the frequency constraints of the first few modes have to be separated. In this example, because of the intrinsic nature of symmetry in the structure, any attempt to separate the fundamental and second natural frequencies in the optimum design failed. To quantify the performance of the algorithm for multiple-frequency constraint problems, the structure was designed using the DM and FM for $\omega_1 = 4$ and $\omega_3 \geq 6$. A minimum weight of 721.597 lbm (327.605 kg) was obtained. The results are given in Tables 3 and 4.

Table 4 Final design results for the natural frequencies (Hz) with different sets of frequency constraints for the 72-bar truss structure

		DM		FM	
Frequency no.	Initial design	$\omega_1 = 4$	$\omega_1 = 4$, $\omega_3 \geq 6$	$\omega_1 = 4$	$\omega_1 = 4$, $\omega_3 \geq 6$
1	3.113	4.000	4.000	4.000	4.000
2	3.113	4.000	4.000	4.000	4.000
3	5.374	5.001	6.000	5.001	6.000
4	9.425	6.505	6.247	6.505	6.247
5	13.189	8.595	9.074	8.595	9.074

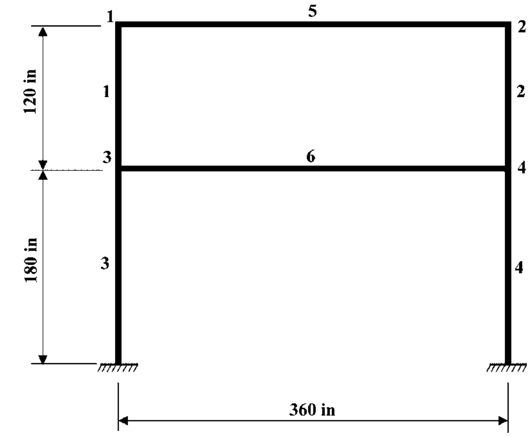

Fig. 4 Six-member frame (two story and one bay).

C. Six-Member Frame (Two-Story and One-Bay)

The six-member frame is illustrated in the Fig. 4. This problem has been studied by Khan and Willmert[15] and McGee and Phan[23] using the optimality criterion method combined with the finite element method based on the displacement formulation.

A uniformly distributed nonstructural mass of 10 lbm/in. (178.740 kg/m) was added to the horizontal members of the frame structure. The density and Young's modulus are 0.28 lbm/in.^{3} (7757 kg/m^3) and 30,000,000 psi (20.7×10^{10}), respectively. The moment of inertia I is empirically related to the area A by the following expressions[15,23]:

$$I = 4.6248A^2, \qquad 0 \leq A \leq 44$$

$$I = 256A - 2300, \qquad 44 < A \leq 88.2813$$

where A is square inches. A constraint (minimum) on the design variables (cross-sectional area of the members) was specified at 7.9187 in.^{2} (51.088 cm^2), and a maximum was set to 88.28 in.^{2} (569.55 cm^2). At the initial design all of the cross-sectional areas are equal to 30 in.^{2} (193.55 cm^2) with an initial mass of 11,088 lbm (5034 kg). First, the structure was analyzed using both the DM and FM giving a fundamental natural frequency of 78.5 rad/s. A minimum mass of 9410.39 lbm (4272.32 kg) was obtained. The final design variables (cross-sectional areas) are different in the DM and FM solutions, suggesting that the optimum solution is not unique. However, the final natural frequencies are the same. Although the optimum results obtained using the FM and DM are different, they resulted in the same optimum mass and same final natural frequency. Therefore, both are optimum solutions. The results are given in the Tables 5 and 6. Khan and Willmert[15] and McGee and Phan[23] report a minimum weight of 9561 and 9815 lbm (4341 and 4456 kg), respectively. To verify that the optimality criterion employed by Refs. 15 and 23 may have produced a local minimum, another simulation was performed, starting with the solution reported in Refs. 15 and 23. This solution process resulted in a design change and converged to a lighter solution of 9410.39 lbm (4272.32 kg). Thus, it is confirmed that the solution in Refs. 15 and 23 does not represent a local minimum.

Table 5 Final design for the cross-sectional areas (cm^2) for different sets of frequency constraints (rad/s) for the six-member frame

Element no.	DM $\omega_1 = 78.5$	DM $\omega_1 = 78.5$, $\omega_2 \geq 180$	FM $\omega_1 = 78.5$	FM $\omega_1 = 78.5$, $\omega_2 \geq 180$
1	215.867	120.556	51.088	206.289
2	51.088	141.802	215.551	62.746
3	51.088	283.870	365.982	138.767
4	367.203	227.761	51.088	297.876
5	51.088	51.088	51.088	51.088
6	253.087	228.549	253.799	256.361
Mass, kg	4272.35	4418.46	4272.32	4365.56
No. of iterations	320	726	258	246
No. of active constraints	4	2	4	3
CPU time, s	15.31	34.29	11.76	11.21

Table 6 Final design results for the natural frequencies (rad/s) for different sets of frequency constraints for the six-member frame

Frequency no.	Initial design	DM $\omega_1 = 78.5$	DM $\omega_1 = 78.5$, $\omega_2 \geq 180$	FM $\omega_1 = 4$	FM $\omega_1 = 78.5$, $\omega_2 \geq 180$
1	69.044	78.500	78.500	78.500	78.500
2	286.840	146.670	220.806	146.668	180.000
3	380.324	268.399	436.420	268.350	371.289
4	476.168	350.723	486.975	350.667	418.804
5	499.720	465.900	540.125	465.780	485.897

The structure was again designed using multiple natural frequency constraints of $\omega_1 = 78.5$ Hz and $\omega_2 \geq 180$. Surprisingly, the optimum mass of 9615.78 lbm (4365.56 kg) using FM and 9732.3 lbm (4418.46 kg) using DM was obtained. As explained before, this specific problem is path dependent, and the slight difference in output results from analyzers (FM and DM) may have caused a different optimum solution. Investigating the final natural frequencies obtained from the DM and the FM, it was revealed that in the FM the inequality constraint is active in the optimum solution, and this is not observed in the solution from the DM. This is the reason for a lighter mass obtained using the FM. For this case the FM performed better computationally than the DM. It can be inferred that for frequency constraint problems the computational time totally depends on the iteration number.

VI. Conclusions

A structural optimization algorithm has been developed to minimize the weight of truss and frame-type structures under single- or multiple-frequency constraints. The integrated force formulation has been used to analyze the dynamics of the structure. The required compatibility matrix in the formulation has been derived directly using a displacement-deformation relation and the single value decomposition technique. Mathematical programming based on the sequential quadratic programming technique has been used to optimize the structure.

The main objective is to investigate the relative performance of the force and displacement methods to design and optimize truss and frame-type structures with frequency constraints. It is concluded that in some problems with multiple-frequency constraints the optimization based on the force method can result in a lighter design. This is not a general case, and these are specific results for the examples presented. Moreover, it is also demonstrated that the sequential quadratic programming method can result in a lighter final optimal design in comparison to the commonly used optimality criterion technique (as shown in the examples). The proposed method has proved to be efficient when analyzing and optimizing truss and beam-type structures.

References

[1]Venkayya, V. B., "Structural Optimization: A Review and Some Recommendations," *International Journal for Numerical Methods in Engineering*, Vol. 13, No. 2, 1978, pp. 203–228.

[2]Mohr, G. A., *Finite Elements for Solids, Fluids, and Optimization*, Oxford Univ. Press, Oxford, 1992, pp. 110–115.

[3]Flurry, C., and Schmit, L. A., Jr., "Dual Methods and Approximation Concepts in Structural Synthesis," NASA CR-3226, Dec. 1980.

[4]Haftka, R. T., and Gurdal Zafer, *Elements of Structural Optimization*, 3rd ed. Kluwer Academic, Norwell, MA, 1992.

[5]Sedaghati, R., Tabarrok, B., Suleman, A., and Dost, S., "Optimization of Adaptive Truss Structures Using Finite Element Force Method Based on Complementary Energy," *Transactions of the Canadian Society for Mechanical Engineering*, Vol. 24, No. 1b, 2000, pp. 263–271.

[6]Sedaghati, R., Tabarrok, B., and Suleman, A., "Integrated Force Method and Optimization of Adaptive Truss Structures," *Proceedings of the International Conference on Computational Engineering and Sciences ICES'2K*, edited by A. Atluri, Vol. 2, 2000, pp. 1598–1603.

[7]Timoshenko, S., *History of Strength of Material*, McGraw–Hill, New York, 1953, pp. 75–78.

[8]Robinson, J., "Automatic Selection of Redundancies in the Matrix Force Method, the Rank Technique," *Canadian Aeronautical Space Journal*, Vol. 11, 1965, pp. 9–12.

[9]Kaneko, L., Lawo, M., and Thierauf, G., "On Computational Procedures for the Force Method," *International Journal for Numerical Methods in Engineering*, Vol. 18, No. 10, 1982, pp.1469–1495.

[10]Patnaik, S. N., "The Integrated Force Method Versus the Standard Force Method," *Computers and Structures*, Vol. 22, No. 2, 1986, pp. 151–163.

[11]Patnaik, S. N., and Joseph, K. T., "Generation of the Compatibility Matrix in the Integrated Force Method," *Computer Methods in Applied Mechanics and Engineering*, Vol. 55, No. 3, 1986, pp. 239–257.

[12]Patnaik S. N., Berke, L., and Gallagher, R. H., "Integrated Force Method Versus Displacement Method for Finite Element Analysis," *Computers and Structures*, Vol. 38, No. 4, 1991, pp. 377–407.

[13]Kiusalaas, J., and Shaw, R. C., "An Algorithm for Optimal Structural Design with Frequency Constraints," *International Journal for Numerical Methods in Engineering*, Vol. 13, No. 2, 1978, pp. 283–295.

[14]Levy, R., and Chai, K., "Implementation of Natural Frequency Analysis and Optimality Criterion Design," *Computers and Structures*, Vol. 10, No. 1–2, 1979, pp. 277–282.

[15]Khan, M. R., and Willmert, K. D., "An Efficient Optimality Criterion Method for Natural Frequency Constrained Structures," *Computers and Structures*, Vol. 14, No. 5–6, 1981, pp. 501–507.

[16]Elwany, M. H. S., and Barr, A. D. S., "Minimum Weight Design of Beams in Torsional Vibration with Several Frequency Constraints," *Journal of Sound and Vibration*, Vol. 62, No. 3, 1979, pp. 411–425.

[17]Tabak, E., and Wright, P. M., "Optimality Criteria Method for Building Frames," *Journal of Structural Division*, Vol. 107, No. 7, 1981, pp. 1327–1342.

[18]Venkayya, V. B., and Tischler, V. A., "Optimization of Structures with Frequency Constraints," *Proceedings of the Computer Methods for Nonlinear Solids and Structural Mechanics Conference*, ASME AMD-54, 1983, pp. 239–259.

[19]Khan, M. R., "Optimality Criterion Techniques Applied to Frames Having General Cross-sectional Relationships," *AIAA Journal*, Vol. 22, No. 5, 1984, pp. 669–676.

[20]Khot, N. S., "Optimization of Structures with Multiple Frequency Constraints," *Computers and Structures*, Vol. 20, No. 5, 1984, pp. 869–876.

[21]Konzelman, C. J., "Dual Methods and Approximation Concepts for Structural Optimization," M.S. Thesis, Dept. of Mechanical Engineering, Univ. of Toronto, Toronto, ON, Canada, Aug. 1986.

[22]Grandhi, R. V., and Venkayya, V. B., "Structural Optimization with Frequency Constraints," *AIAA Journal*, Vol. 26, No. 7, 1988, pp. 858–866.

[23]McGee, O. G., and Phan, K. F., "A Robust Optimality Criteria Procedures for Cross-Sectional Optimization of Frame Structures with Multiple Frequency Limits," *Computers and Structures*, Vol. 38, No. 5–6, 1991, pp. 485–500.

[24]Khot, N. S., and Kamat, M. P., "Minimum Weight Design of Truss Structures with Geometric Nonlinear Behavior," *AIAA Journal*, Vol. 23, No. 1, 1985, pp. 139–144.

[25]Sedaghati, R., and Tabarrok, B., "Optimum Design of Truss Structures Undergoing Large Deflections Subject to a System Stability Constraint," *International Journal for Numerical Methods in Engineering*, Vol. 48, No. 3, 2000, pp. 421–434.

[26]Golub, G. H., and Van Loan, C. F., *Matrix Computations*, 3rd ed., Johns Hopkins Univ. Press, Baltimore, MD, 1996, pp. 203–210.

[27]Coleman, T., Branch, M. A., and Grace, A., *Optimization Toolbox-For Use with Matlab*, User's Guides, Version 2, The MathWorks, Inc., 1999.

[28]Powell, M. J. D., "A Fast Algorithm for Nonlinearly Constrained Optimization Calculations," *Numerical Analysis*, edited by G. A. Watson, Vol. 630, Lecture Notes in Mathematics, Springer-Verlag, 1978.

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